Einstein constraints on compact n dimensional manifolds.

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Abstract

We give a general survey of the solution of the Einstein constraints by the conformal method on n dimensional compact manifolds. We prove some new results about solutions with low regularity (solutions in H_2 when n=3), and solutions with unscaled sources.

Dedicated to Vincent Moncrief, a great scientist and friend.

1 Introduction.

The geometric (physical) initial data on an n-dimensional manifold M for the Einstein equations are $\bar{g}$ a properly riemannian metric and K a symmetric 2-tensor. They cannot be arbitrary, they must satisfy the constraints, which are the Gauss-Codazzi equations linking the metric $\bar{g}$ induced on M by the space time metric g, and the extrinsic curvature K of M as submanifold imbedded in the space time (V, g), with the value on M of the Ricci tensor of g.

These constraints read as equations on M. They are the so called hamiltonian constraint (we choose units such that $8\pi G = 1$):

$$R(\bar{g}) - K.K + (trK)^2 = 2\rho, \tag{1.1}$$

and the momentum constraint:

$$\bar{\nabla}.K - \bar{\nabla} trK = J. \tag{1.2}$$

ρ is a scalar and J a vector on M determined by the stress energy tensor of the sources and the lapse N. In a Cauchy adapted frame, where the equation of M in V is $x^0 \equiv t$ =constant, one has:

$$J_i = NT_i^0, \quad \rho = N^2 T^{00}$$

with $\rho \geq 0$ if the sources satisfy a positive energy condition.

In this article we use the standard method[1] to solve the constraints, that is the conformal method, initiated by Lichnerowicz and developped by CB and York. We treat the cosmological case[2], that is we suppose the manifold M to be compact (without boundary). We consider arbitrary dimensions and we lower the regularity previously obtained for the solutions: we obtain large classes of solutions in H_2 when $n = 3$. We prove also miscellaneous new results. There is certainly still room for improvements.

2 The conformal method.

A riemannian metric γ is arbitrarily chosen on the manifold M, together with a scalar function τ. The physical metric $\bar{g}$ and second fundamental form K are then defined by[3]:

$$\bar{g}_{ij} \equiv \varphi^{\frac{4}{n-2}} \gamma_{ij}, \quad K^{ij} \equiv \varphi^{\frac{-2n(n+2)}{n-2}} A^{ij} + \frac{1}{n} \bar{g}^{ij} \tau, \tag{2.1}$$

with A^{ij} a traceless symmetric tensor. The sources are supposed to be known, they split into York scaled and unscaled ones, they are given as a pair of non

[1] The constraints were first formulated as an elliptic system in CB 1957, explained also in CB 1962, through harmonic spacetime coordinates. The decomposition of the constraints through the conformal method into a linear system and an elliptic semi linear equation was obtained for zero trace K and zero momentum sources by Lichnerowicz 1944. An elliptic formulation for the linear system was given in CB 1970. The decomposition of the system of constraints was generalized to constant trace K and scaled momentum by York 1972, who obtained an elliptic system by the now universally used conformal splitting. Improved variants of this splitting, useful for numerical computations, have been given by York 1999 and Pfeister and York 2002.

[2] The asymptotically euclidean case (isolated systems) is considered in CB- Isenberg-York 2000. Results with similar low regularity can be proved by analogous methods. They are being obtained independently by D. Maxwell who also treats the case of asymptotically euclidean manifolds with boundary..

[3] In the case n=2 the powers of φ are to be replaced by exponentials (see Moncrief 1986).

negative scalars ρ_1, ρ_2, and a pair of vectors J_1, J_2. The physical sources are:

$$q \equiv \varphi^{-\frac{2(n+1)}{n-2}}\rho_1 + \rho_2, \quad J \equiv \varphi^{-\frac{2(n+1)}{n-2}} J_1 + J_2. \tag{2.2}$$

One denotes by D, and by Δ_γ, the covariant derivative, and the Laplace operator, in the metric γ.

The Hamiltonian constraint becomes the Lichnerowicz equation:

$$\mathcal{H} \equiv \Delta_\gamma \varphi - f(., \varphi) = 0, \tag{2.3}$$

$$f(., \varphi) \equiv r\varphi - a\varphi^{-\frac{3n-2}{n-2}} - q_1 \varphi^{-\frac{n}{n-2}} + (b - q_2)\varphi^{\frac{n+2}{n-2}},$$

with (a dot is the scalar product in γ)

$$r \equiv \frac{n-2}{4(n-1)} R(\gamma) \quad a \equiv \frac{n-2}{4(n-1)} A.A \geq 0, \quad b \equiv \frac{n-2}{4n}\tau^2 \geq 0,$$

$$q_i \equiv \frac{n-2}{2(n-1)}\rho_i \geq 0.$$

The momentum constraint becomes (indices raised with γ):

$$\mathcal{M}^i \equiv D_j A^{ij} - \{\frac{n-1}{n}\varphi^{\frac{2n}{n-2}}\partial^i \tau + \varphi^{\frac{2(n+2)}{n-2}} J_2 + J_1\} = 0, \tag{2.4}$$

In the original conformal formulation one sets:

$$A^{ij} \equiv B^{ij} + (\mathcal{L}_{\gamma,conf} X)^{ij} \tag{2.5}$$

with B^{ij} an arbitrary given traceless tensor, and $\mathcal{L}_{\gamma,conf}X$ the conformal Lie derivative of a vector X to be determined, that is:

$$(\mathcal{L}_{\gamma,conf}X)_{ij} \equiv D_i X_j + D_j X_i - \frac{2}{n}\gamma_{ij} D_h X^h. \tag{2.6}$$

We will call the system 2.3 and 2.4, like Isenberg and Moncrief, the LCBY equations. The unknowns are φ and X.

Remark 2.1 *In the thin sandwich formulation (York 1999), which makes the conformal invariance of the constraints (when τ =constant) more transparent and is useful in numerical computations, one sets:*

$$A^{ij} \equiv (2N)^{-1}\{-u^{ij} + (\mathcal{L}_{\gamma,conf}\beta)^{ij}\} \tag{2.7}$$

with N an arbitrary, positive, scalar, and u^{ij} an arbitrary given traceless tensor. The momentum contraint in this formulation reads, for general n:

$$\mathcal{M}^i \equiv D_j\{(2N)^{-1}(\mathcal{L}_{\gamma,conf}\beta)^{ij}\} - F^i_{TS}(.,\varphi) = 0, \tag{2.8}$$

$$F^i_{TS}(.,\varphi) \equiv D_j\{(2N)^{-1}u^{ij}\} + \frac{n-1}{n}\varphi^{2n/n-2}\partial^i\tau + \varphi^{2(n+2)/(n-2)}J_2 + J_1.$$

The unknowns are now β and φ. The mathematical results are essentially the same as in the original formulation which we use here for simplicity of notations.

Functional spaces.

To have good functional spaces it is convenient to introduce a given smooth properly riemannian metric e on M.

The Sobolev spaces W^p_s on (M,e) are defined as closures of spaces of smooth tensor fields in the norm

$$||f||_{W^p_s} \equiv \{\int_M \sum_{0\leq k\leq s} |\partial^k f|^p \mu_e\}^{\frac{1}{p}}, \tag{2.9}$$

where ∂, $|.|$ and μ_e are the covariant derivative, the pointwise norm and the volume element in the metric e. A metric γ is said to belong to M^p_σ if it is properly riemannian and $\gamma \in W^p_\sigma$. We will always suppose that $\sigma > \frac{n}{p}$, then $\gamma \in M^p_\sigma$ is continuous on M as well as its contrevariant associate $\gamma^\#$, and M^p_σ is an open set in W^p_σ. The volume element μ_γ of γ is equivalent to μ_e.

3 Solution of the momentum constraint.

The momentum constraint 2.4 reads:

$$(\Delta_{\gamma,conf}X)^i \equiv D_j(\mathcal{L}_{\gamma,conf}X)^{ij} = F^i, \tag{3.1}$$

$$F^i \equiv \frac{n-1}{n}\varphi^{2n/(n-2)}\partial^i\tau + \varphi^{2(n+2)/(n-2)}J_2 + J_1 - D_iB^{ij} \tag{3.2}$$

where B is a traceless symmetric 2-tensor. When γ, properly riemannian, τ, J_1, J_2, and B are given and φ is known 3.1 is a linear system for the vector X.

Lemma 3.1 *If $\gamma \in M^p_{2+s}$, $s \geq 0$, $p > \frac{n}{2}$ the vector F belongs to W^p_s, as soon as $\varphi \in W^p_{2+s}$, $D\tau, J \in W^p_s$ and $B \in W^p_{s+1}$.*

PROOF. The Sobolev multiplication theorem. ■

Theorem 3.2 *The momentum constraint 3.1 with $\gamma \in M^p_{2+s}$, $p > \frac{n}{2}$, $s \geq 0$ has a solution $X \in W^p_{2+s}$, if $F \in W^p_s$ and is L^2 orthogonal to the space of conformal Killing (CK) vector fields of (M, γ).*

The solution is determined up to addition of a CK vector. It is unique if we impose that it be L^2 orthogonal to CK vectors. There exists then a constant $C_\gamma > 0$ depending only on γ such that

$$\parallel X \parallel_{W^p_{s+2}} \leq C_\gamma \parallel F \parallel_{W^p_s} . \tag{3.3}$$

PROOF. We show that the operator $\Delta_{\gamma,conf}$ satisfies the theorems 7.2 and 7.4 of the appendix A.

1. The operator is elliptic: its principal symbol at x, with $\xi \in T_xM$ is the linear mapping from covariant vectors b into covariant vectors a given by

$$\xi^i\xi_i b_j + \xi^i\xi_j b_i - \frac{2}{n}\xi_j\xi^k b_k = a_j . \tag{3.4}$$

This linear mapping is an isomorphism if $\xi \neq 0$: its characteristic determinant is computed to be ($\gamma^\#$ is the contrevariant tensor associated with γ) :

$$D(\xi) \equiv (\xi^i\xi_i)^n(1 - \frac{1}{n}) \equiv \gamma^\#(\xi, \xi) > 0 \text{ if } \xi \neq 0. \tag{3.5}$$

2. The coefficients of the operator $\Delta_{\gamma,conf}$ are of the type :

$$a_2 \tilde{=} \gamma^\#, \quad a_1 \tilde{=} \gamma^\# \partial\gamma, \quad a_0 \tilde{=} \gamma^\# \partial\gamma\partial\gamma + \gamma^\# \partial^2\gamma \tag{3.6}$$

they satisfy the hypothesis of the theorem 7.2 of the appendix A if

$$a_2 \in M^p_2, \quad a_1 \in W^p_1, \quad a_0 \in L^p \text{ with } p > \frac{n}{2} . \tag{3.7}$$

Indeed if $\gamma \in M^p_2, p > \frac{n}{2}$, then also $\gamma^\# \in M^p_2 \subset C^{0,\alpha}$, while $\partial\gamma\partial\gamma \in W^p_1 \times W^p_1 \subset L^p$, hence $a_1 \in W^p_1$, and $a_0 \in L^p$.

2. Smooth vectorfields X and Y with smooth γ satisfy, on a compact manifold, the following identity:

$$\int_M Y_j(\Delta_{\gamma,conf}X)^j \mu_\gamma \equiv \int_M Y_j D_i \left(D^i X^j + D^j X^i - \frac{2}{n}\gamma^{ij} D_k X^k \right) \mu_\gamma \equiv \tag{3.8}$$

$$-\int_M \left(D^iY^j + D^jY^i - \frac{2}{n}\gamma^{ij}D_kY^k\right)\left(D_iX_j + D_jX_i - \frac{2}{n}\gamma_{ij}D_lX^l\right)\mu_\gamma$$

Under the hypothesis made on p the operator $\Delta_{\gamma,conf}$ is a continuous mapping $W_2^p \to L^p$, hence $Y.\Delta_{\gamma,conf}X$ is a continuous mapping $W_2^p \times W_2^p \to L^1$. The conformal Killing operator is of the form $\partial + \gamma^{\#}\partial\gamma$, it is a continuous mapping from W_2^p into W_1^p, and by the Sobolev embedding it holds that $W_1^p \times W_1^p \subset L^1$ as soon as $p \geq \frac{2n}{n+2}$, a fortiori if $p > \frac{n}{2}$.

The continuity of all the considered embeddings permits the passage to the limit which proves the identity 3.8 in our low regularity case. This identity implies, making $X = Y$, $\Delta_{\gamma,conf}X = 0$, the following one:

$$(\mathcal{L}_{\gamma,conf}X)_{ij} \equiv D_iX_j + D_jX_i - \frac{2}{n}\gamma_{ij}D_lX^l = 0. \tag{3.9}$$

The identity 3.8 also shows that $\Delta_{\gamma,conf} - k : W_2^p \to L^p$ is injective, hence an isomorphism, if k is a strictly positive number. Its inverse is then a compact operator, hence $\Delta_{\gamma,conf}$ is a Fredholm operator, $\Delta_{\gamma,conf}X = F$ has a solution $X \in W_2^p$ iff F is orthogonal to the kernel of the adjoint operator, which is $\Delta_{\gamma,conf}$ itself. The solution is unique in the Banach space of W_2^p vectors orthogonal to CK vectors.

The proof of higher regularity for more regular coefficients, and of the inequality 3.3 is obtained by derivating the equation. ■

Remark 3.3 *If the Ricci tensor*[4] *of γ, with components ρ_{ij}, is negative definite the manifold (M, γ) admits no conformal Killing fields. Indeed the equality $\Delta_{\gamma,conf}X = 0$ implies, using the Ricci identity,*

$$D^iD_iX_j + (1 - \frac{2}{n})D_jD^iX_i + \rho_{jl}X^l = 0, \tag{3.10}$$

which implies on a compact manifold

$$\int_M \{-D^iX^jD_iX_j - (1 - \frac{2}{n})D_jX^jD^iX_i + \rho_{jl}X^lX^j\}\mu_\gamma = 0, \tag{3.11}$$

from which follows $X \equiv 0$ if Ricci(γ) is negative definite and $n \geq 2$.

Remark 3.4 *Even when X is not unique (i.e. if (M, γ) admits a C.K vector field) the tensor A is determined uniquely through the formula 2.5.*

[4]The hypothesis $\gamma \in M_2^p$, p>n/2 implies that the Riemann and Ricci tensors of γ are in L^p.

Lemma 3.5 *When $D\tau \equiv 0$ and $J_2 \equiv 0$, then F is L^2 orthogonal to CK vector fields if and only if it is so of J_1.*

PROOF. If $D\tau \equiv 0$ and $J_2 \equiv 0$, then F reduces to the sum of J_1 and the divergence of a traceless symmetric tensor. Such divergences are L^2 orthogonal to CK vector fields. This fact well known in a smoother case results from the identity, still valid under our hypotheses

$$\int_M D_i B^{ij} X_j \mu_\gamma = -\int_M B^{ij} (\mathcal{L}_{\gamma,conf} X)_{ij} \mu_\gamma. \tag{3.12}$$

■

If $D\tau \not\equiv 0$ or $J_2 \not\equiv 0$ (unscaled momentum sources) there seems to be at present no result for manifolds (M, γ) admitting CK vector fields.

4 Lichnerowicz equation.

It reads

$$\Delta_\gamma \varphi = f(., \varphi) \equiv r\varphi - a\varphi^{-\frac{3n-2}{n-2}} - q_1 \varphi^{-\frac{n}{n-2}} + (b - q_2)\varphi^{\frac{n+2}{n-2}}, \tag{4.1}$$

we consider in this section that the coefficients a, q_i, b are given non negative functions on M.

4.1 The Yamabe properties.

A theorem conjectured by Yamabe and proved by Trudinger, Aubin, Schoen is interesting to classify solutions obtained by the conformal method.

definition 4.1 *The functional*

$$J_\gamma(\varphi) \equiv \frac{\int_M (k_n^{-1} |\nabla\varphi|^2 + R(\gamma)\varphi^2)\mu_\gamma}{(\int_M \varphi^{2n/(n-2)} \mu_\gamma)^{(n-2)/n}}. \tag{4.2}$$

defined for every $\varphi \in H_1 \subset L^{\frac{2n}{n-2}}$, $\varphi \not\equiv 0$ and $\gamma \in M_2^p$ $p > \frac{n}{2}$, is called the Yamabe functional.

The functional $J_q(\varphi)$ admits an infimum for $\varphi \in W_2^p$ (dense in H_1 if $p > \frac{n}{2}$), and $\varphi \not\equiv 0$, because it is bounded below as shown by the inequality:

$$\int_M (k_n^{-1}|\nabla\varphi|^2 + R(\gamma)\varphi^2)\mu_\gamma \geq -|\int_M R(\gamma)\varphi^2\mu_\gamma| \geq -||R(\gamma)||_{L^{\frac{n}{2}}}||\varphi^2||_{L^{\frac{n}{n-2}}} \tag{4.3}$$

This infimum μ:

$$\mu \equiv \underset{\varphi\in W_2^p, \varphi\not\equiv 0}{Inf} J_\gamma(\varphi) \equiv \underset{\varphi\in W_2^p, \varphi\geq 0, \varphi\not\equiv 0}{Inf} J_\gamma(\varphi), \tag{4.4}$$

depends only on the conformal class of γ, it is called the Yamabe invariant. The manifolds (M, γ) are split into 3 **Yamabe classes** according to the sign of μ.

definition 4.2 *The manifold (M, γ) is said to be in the negative Yamabe class if $\mu < 0$, in the zero Yamabe class if $\mu = 0$, in the positive Yamabe class if $\mu > 0$.*

Remark 4.3 *It is known (Kazdan and Warner (1975, 1985) that a sufficient condition for (M, γ) to be of negative Yamabe class is $\int_M R(\gamma)\mu_\gamma < 0$, and that every compact manifold of dimension $n \geq 3$ admits a metric in the negative Yamabe class. On the other hand, not all manifolds (Lichnerowicz 1963) can support metrics in the positive or zero Yamabe class.*

Possible topologies of manifolds of various Yamabe type are reviewed in Fisher and Moncrief 1994.

The following theorem conjectured by Yamabe has been proved in an increasing number of cases by Trudinger and Aubin and finally completed by Schoen, by showing that the infimum is attained by a function φ_m.

Theorem 4.4 *Let M be a compact smooth manifold. Any smooth riemannian metric γ on M is conformal to a metric with constant scalar curvature, the sign of which is a conformal invariant.*

Remark 4.5 *The Yamabe theorem says that the minimum φ_m, hence γ_m, is unique if $\mu < 0$, but not necessarily so if $\mu > 0$.*

The Yamabe invariant is defined as soon as $\gamma \in M_2^p$, $p > \frac{n}{2}$, since then $R(\gamma) \in L^p$, and $\varphi \in W_2^p \subset C^0$. Checking of the different steps of the proof shows that the existence of $\varphi_m \in W_2^p$, $\varphi_m > 0$ can be extended with no real difficulty to these low regularity metrics in the cases $\mu < 0$ and $\mu = 0$. Difficulties for this generalization appear in the case $\mu > 0$ and specially when $n = 3$. The proof given by Schoen uses indeed the geodesic balls and the Green function which require at least a $C^{1,1}$ metric, implied only by $\gamma \in W_{2+s}^p$, $p > \frac{n}{2}$, if $s \geq 2$. The weaker theorem proved by Yamabe[5] extends to $\gamma \in M_2^p$, $p > \frac{n}{2}$ and will be sufficient for our use.

Theorem 4.6 *Let (M, γ) be a compact riemannian manifold with $\gamma \in M_2^p$, $p>\frac{n}{2}$. Then:*

1. If γ is in the negative Yamabe class it is conformal to a metric with scalar curvature -1.

2. If γ is in the zero Yamabe class it is conformal to a metric with scalar curvature 0.

3. If γ is in the positive Yamabe class if it is conformal to a metric with continuous and positive scalar curvature $r \geq 1$.

4.2 Non existence and uniqueness.

definition 4.7 *Let $f \geq 0$ be a function defined almost everywhere on M. We say that*[6] *$f \not\equiv 0$ if there is an open set of M where $Inf\ f > 0$.*

Theorem 4.8 *(non existence) The Lichnerowicz equation admits no solution $\varphi > 0$, $\varphi \in W_2^p$, $p > \frac{n}{2}$, on a compact manifold (M,γ), $\gamma \in M_2^p$, $a, b, q \in L^1$ if either:*

1. $r \leq 0$, $b - q_2 \leq 0$, and $-r + a + q_1 - b + q_2 \not\equiv 0$.

2. $r \geq 0$, $a \equiv q_1 \equiv 0, b - q_2 \geq 0$ and $r + b - q_2 \not\equiv 0$.

Proof. The integral of $\Delta_{\gamma_n}\varphi_n$ with respect to the volume element of γ_n is equal to zero on a compact manifold M if γ_n is C^1 and φ_n is C^2, by the Stokes formula. We approximate the given γ and φ in $W_2^p \subset C^0$ by such γ_n and φ_n. Denote by μ_{γ_n} and μ_γ the volume elements of respectively γ_n and γ, μ_{γ_n} tends in C^0 to μ_γ. The Sobolev embedding and multiplication theorems

[5]See for instance Aubin p.127.

[6]If a is continuous the definition coincides with the usual notation.

show that $\Delta_\gamma \varphi(\mu_e)^{-1}\mu_\gamma - \Delta_{\gamma_n}\varphi(\mu_e)^{-1}\mu_{\gamma_n}$ tends to zero in L^1 (see analogous proofs in the appendix), hence the integral of $\Delta_\gamma\varphi$ on (M,γ) is zero.

On the other hand, $f(x,\varphi)$ is integrable on M since $\varphi \in C^0$ and $\varphi > 0$ and its integral cannot vanish under the condition 1 [respectively 2] which implies that $f(x,\varphi) \leq 0$ [respectively ≥ 0] and is strictly negative [positive] on an open set of M. ■

The geometrical origin of the Lichnerowicz equation leads to a general uniqueness theorem, independant of the sign of r, hence of the Yamabe class of γ.

Theorem 4.9 *(uniqueness[7]) The Lichnerowicz equation 4.1 on* (M,γ), $\gamma \in M_2^p, p > \frac{n}{2}$, *has at most one positive solution* $\varphi \in W_2^p$, *if* $a, q_i, b \in L^1$, $a \geq 0$, $q_1 \geq 0$, $b - q_2 \geq 0$, *and* $a+\ q_1+\ b - q_2 \not\equiv 0$ *on* M.

Proof. Suppose it admits two solutions $\varphi_1 > 0$ and $\varphi_2 > 0$. The following identity holds:

$$\Delta_{\varphi_2^{4/(n-2)}\gamma}(\varphi_1\varphi_2^{-1}) - r(\varphi_2^{4/(n-2)}\gamma)(\varphi_1\varphi_2^{-1}) \equiv -(\varphi_1\varphi_2^{-1})^{(n+2)/(n-2)} r(\varphi_1^{4/(n-2)}\gamma) \tag{4.5}$$

Since φ_1 is a solution of 4.1 we have an equality of the form:

$$\begin{aligned} r(\varphi_1^{4/(n-2)}\gamma) &\equiv -\varphi_1^{-(n+2)/(n-2)}\{\Delta_\gamma\varphi_1 - \varphi_1 r(\gamma)\} \\ &= \varphi_1^{-(n+2)/(n-2)}\{a\varphi_1^P + q_1\varphi^{P_1} - \varphi^Q(b-q_2)\} \end{aligned}$$

and an analogous equation for $r(\varphi_2^{4/(n-2)}\gamma)$. Inserting in the previous equation gives the equation:

$$\Delta_{\varphi_2^{2q}\gamma}(\varphi_1\varphi_2^{-1} - 1) - \lambda\{(\varphi_1\varphi_2^{-1} - 1\} = 0 \tag{4.6}$$

with

$$\begin{aligned} \lambda \equiv\ & a\varphi_1^{(-3n+2)/(n-2)}\varphi_2^{-(n+2)/(n-2)}\frac{(\varphi_1\varphi_2^{-1})^{4(n-1)/(n-2)} - 1}{\varphi_1\varphi_2^{-1} - 1} + \\ & q_1\varphi_1^{-n/(n-2)}\varphi_2^{-(n+2)/(n-2)} + (b-q_2)\varphi_1\varphi_2^{-1}\frac{(\varphi_1\varphi_2^{-1})^{4/(n-2)} - 1}{\varphi_1\varphi_2^{-1} - 1} \end{aligned}$$

The fractions with denominator $\varphi_1\varphi_2^{-1} - 1$ are continuous and positive functions on M if it is so of the $\varphi' s$, because of the mean function theorem which

[7] First proved (case n=3, smooth coefficients) by Araki, see CB 1962.

says that if a and α are two positive numbers there exist a number b such that

$$\frac{a^{\alpha}-1}{a-1}=\alpha b^{\alpha-1}, \quad b\in[a,1] \text{ if } 0<a\leq 1, \qquad b\in[1,a] \text{ if } a\geq 1. \tag{4.7}$$

Therefore λ is an integrable function on M, depending on φ_1, φ_2, a, q_1, q_2 and b, with $\lambda \geq 0$ and $\lambda \not\equiv 0$ on M if $a+q_1+b-q_2\geq 0$ and $\not\equiv 0$. The equation implies therefore[8] $\varphi_1\varphi_2^{-1}-1\equiv 0$. ■

4.3 Existence theorem, scaled sources in L^{∞}.

The results can be obtained, as in the case $n=3$, by using the Leray Schauder degree, CB 1972 (unscaled sources), O'Murchada and York 1974 (scaled sources), or a constructive method, Isenberg 1987. Both methods use sub and supersolutions. The following theorem is an extension of results obtained before with higher regularity, and for $n=3$ (see reviews in CB-York 1980, Isenberg 1995). In the theorems some coefficients are supposed to be in L^{∞}, and not just in L^p, because we look for constant sub and supersolutions. A new result is an extension to cases where $\tau^2\not\equiv 0$ in the cases of negative or zero Yamabe class.

Theorem 4.10 *The Lichnerowicz equation with scaled sources on a compact n-manifold (M,e) with given riemannian metric $\gamma\in M_2^p$, $p>\frac{n}{2}$, and a, b, $q_1\in L^{\infty}$, with unknown the conformal factor φ, admits a solution $\varphi>0$, $\varphi\in W_2^p$ if:*

1. (M,γ) is in the positive Yamabe class and $\underset{M}{Inf}(a+q_1)>0$.

2. (M,γ) is in the zero Yamabe class, $\underset{M}{Inf}\tau^2>0$ and $\underset{M}{Inf}(a+q_1)>0$.

3.(M,γ) is in the negative Yamabe class and $\underset{M}{Inf}\tau^2>0$.

Proof. The general existence theorem 8.2 shows that, under the hypothesis made on the coefficients, the Lichnerowicz equation admits a solution $\varphi>0$, $\varphi\in W_2^p$, if we can find constants ℓ and m such that on M:

$$f(.,\ell)\leq 0, \quad f(.,m)\geq 0 \quad \text{with} \quad 0<\ell\leq m. \tag{4.8}$$

We have:

[8] See appendix A, lemma 7.6.

$$f(x,y) \equiv y^{-\frac{3n-2}{n-2}} h(x,y), \tag{4.9}$$

$$h(x,y) \equiv b(x) y^{\frac{4n}{n-2}} + r(x) y^{\frac{4(n-1)}{n-2}} - q_1(x) y^{\frac{2(n-1)}{n-2}} - a(x) \tag{4.10}$$

There exist $\ell > 0$ and $m \geq \ell$ satisfying 4.8 if they satisfy the same inequalities with h. We have

$$h'_y \equiv \frac{2}{n-2} \tilde{h}(x,y) y^{\frac{n}{n-2}}, \quad \tilde{h}(x,y) \equiv 2nby^{\frac{2n+2}{n-2}} + 2(n-1)ry^{\frac{2n-2}{n-2}} - (n-1)q_1. \tag{4.11}$$

$$\tilde{h}'_y \equiv \frac{8}{n-2} \{n(n+1)by^{\frac{4}{n-2}} + (n-1)^2 r\} y^{\frac{n}{n-2}}. \tag{4.12}$$

The function $\tilde{h}'_y(x,.)$ has at most one zero $Y_0(x) > 0$; the function $\tilde{h}(x,.)$ starts from $-(n-1)q_1(x) \leq 0$ for $y = 0$. We examine the various Yamabe classes.

- 1. (M, γ) is of positive Yamabe class, we can suppose r continuous and $r(x) \geq 1$ at every point $x \in M$. Then for any given $x \in M$ the function $h'_y(x,.)$ is a non decreasing function of $y \geq 0$, non negative as soon as

  $$y \geq m(x), \quad \text{with } \; m(x) \text{ a number such that } \quad m(x)^{\frac{2n-2}{n-2}} \geq \frac{1}{2} q_1(x). \tag{4.13}$$

  The function $h(x,.)$ of y is non decreasing when $y \geq m(x)$, it is non negative if it is so for $y = m(x)$. A sufficient condition is that, since $b \geq 0$,:

  $$m(x)^{\frac{4(n-1)}{n-2}} \geq q_1(x) m(x)^{\frac{2(n-1)}{n-2}} + a(x) \tag{4.14}$$

  which is implied for instance by the sufficient condition

  $$m(x) \geq Max\{1, a(x) + q_1(x)\} \tag{4.15}$$

Any number m (independent of x) such that

$$m \geq \underset{M}{Sup}(a + q_1), \quad m \geq 1 \tag{4.16}$$

will satisfy the condition 4.15 almost everywhere on M, hence will be a supersolution.

The function $h(x,.)$ is non positive for some $\ell(x)$ if

$$b(x)\ell(x)^{\frac{4n}{n-2}} + r(x)\ell(x)^{\frac{4(n-1)}{n-2}} - q_1(x)\ell(x)^{\frac{2(n-1)}{n-2}} - a(x) \leq 0 \tag{4.17}$$

A sufficient condition is

$$\ell(x) \leq 1, \quad (b(x)+r(x))\ell^{\frac{4(n-1)}{n-2}} \leq (q_1(x)+a(x))\ell^{\frac{2(n-1)}{n-2}} \tag{4.18}$$

a sufficient condition is therefore:

$$\ell(x) \leq 1, \quad \ell(x) \leq (b(x)+r(x))^{-1}(q_1(x)+a(x)) \tag{4.19}$$

Therefore if $\underset{M}{Inf}(a+q_1) > 0$ any number ℓ such that

$$0 < \ell \leq \{\underset{M}{Sup}(b+r)\}^{-1}\{\underset{M}{Inf}(q_1+a)\}, \quad \ell \leq 1. \tag{4.20}$$

is a constant subsolution,and it is possible to choose ℓ and m such that

$$0 < \ell \leq m. \tag{4.21}$$

- 2. (M,γ) is of zero Yamabe class, we suppose $r(\gamma) = 0$. Then:

$$h(x,y) \equiv b(x)y^{\frac{4n}{n-2}} - q_1(x)y^{\frac{2(n-1)}{n-2}} - a(x) \tag{4.22}$$

  There exist $\ell > 0$ and $m \geq \ell$ satisfying 4.8 if they satisfy the same inequalities with h. We have

$$h'_y \equiv \frac{2}{n-2}\tilde{h}(x,y)y^{\frac{n}{n-2}}, \quad \tilde{h}(x,y) \equiv 2nby^{\frac{2n+2}{n-2}} - (n-1)q_1. \tag{4.23}$$

$$\tilde{h}'_y \equiv \frac{4}{n-2}n(n+1)by^{\frac{n+4}{n-2}}. \tag{4.24}$$

There exist constant supersolutions $0 < \ell \leq m$ if $\underset{M}{Inf}(a+q_1,b) > 0$ and $\underset{M}{Sup}(a+q_1,b) < +\infty$. They are then chosen such that:

$$0 < \ell \leq Min\{1, \frac{\underset{M}{Inf}(a+q_1)}{\underset{M}{Sup} b}\}, \quad m \geq Max\{1, \frac{\underset{M}{Sup}(a+q_1)}{\underset{M}{Inf} b}. \tag{4.25}$$

- 3. (M, γ) is of negative Yamabe class, we take $r = -1$..

$$h(x, y) \equiv b(x)y^{\frac{4n}{n-2}} - y^{\frac{4(n-1)}{n-2}} - q_1(x)y^{\frac{2(n-1)}{n-2}} - a(x) \tag{4.26}$$

$$h'_y \equiv \frac{2}{n-2}\tilde{h}(x, y)y^{\frac{n}{n-2}}, \quad \tilde{h}(x, y) \equiv 2nby^{\frac{2n+2}{n-2}} - 2(n-1)y^{\frac{2n-2}{n-2}} - (n-1)q_1. \tag{4.27}$$

$$\tilde{h}'_y \equiv \frac{4}{n-2}\{n(n+1)by^{\frac{4}{n-2}} - (n-1)^2\}y^{\frac{n}{n-2}}. \tag{4.28}$$

If $b(x) > 0$ then $\tilde{h}(x, .)$ starts from a non positive value, decreases until $y = Y_0(x)$,

$$Y_0(x) = \{\frac{(n-1)^2}{n(n+1)b(x)}\}^{\frac{n-2}{4}}, \tag{4.29}$$

increases afterwards up to $+\infty$, it has therefore one positive zero $Y(x) > Y_0(x)$. Hence $h(x, .)$ starts from a non positive value, decreases until $y = Y(x)$ then increases up to $+\infty$. We see that if $Inf\ b > 0$ on M one can always find ℓ and m satisfying 4.4 by choosing ℓ and m such that:

$$0 < \ell \leq \underset{M}{Inf} Y(x), \qquad m \geq \underset{M}{Sup} Y(x). \tag{4.30}$$

■

Sub and supersolutions do not need to be constants: we will obtain the following theorem by looking for non constant sub or supersolutions. It replaces inequalities to be satisfed on M by inequalities on an open set of M.

Corollary 4.11 *The theorem 4.10 holds if*

1. The hypothesis $\underset{M}{Inf}(a+q_1) > 0$ is replaced by $a+q_1 \not\equiv 0$ for the positive or zero Yamabe classes.

2. The hypothesis $\underset{M}{Inf}\tau^2 > 0$ is replaced by:

a. $\tau^2 \not\equiv 0$ for the zero Yamabe class.

b. $Inf\ \tau^2 > 0$ on a sufficiently large subset of M for the negative Yamabe class.

PROOF. 1. We show as in Isenberg 1987 that the condition $\underset{M}{Inf}(a+q_1) > 0$ is superfluous for the construction of a non constant φ_-, by constructing a constant subsolution for a conformally transformed equation. Suppose that $a + q_1 \not\equiv 0$, i.e. there is an open subset $U \subset M$ where $\underset{U}{Inf}(a + q_1) > 0$. The complementary set $M - U$ is contained in a proper open subset V which we can take with smooth boundary ∂V. We can construct a metric γ' conformal to γ whose scalar curvature is stricly negative in V, as follows. Set $\gamma' = \theta^{4/(n-2)}\gamma$, the formula for the conformal transformation of the scalar curvature gives:

$$r(\gamma') \equiv \theta^{-\frac{n+2}{n-2}}(-\Delta_\gamma\theta + r(\gamma)\theta). \tag{4.31}$$

Take for $\theta \in W_3^p$ the unique strictly positive solution in V of the linear equation and boundary condition, with k some positive constant,

$$\Delta_\gamma\theta - k\theta = 0, \quad \theta\,|_{\partial V} = 1. \tag{4.32}$$

These two equations imply that:

$$r(\gamma') = \theta^{-\frac{n+2}{n-2}}(r(\gamma) - k).$$

We take $k > \underset{M}{Sup}r(\gamma)$ we have then $r(\gamma') < 0$, with $\underset{V}{Inf}|r(\gamma')| > 0$, in V. We choose on M a smooth function $\theta > 0$ equal on V to the previously determined θ. The function $\varphi' = \theta^{-1}\varphi$ satisfies the following equation, where $a' = a\theta^{-\frac{4n}{n-2}}$, $q_1' = q_1\theta^{-\frac{4n}{n-2}}$, hence $\inf\limits_{M-V}(a' + q_1') > 0$:

$$\Delta_{\gamma'}\varphi' - r(\gamma')\varphi' + a'\varphi'^{-\frac{3n-2}{n-2}} + q_1'\varphi'^{-\frac{n}{n-2}} - b\varphi'^{\frac{n+2}{n-2}} = 0.$$

The number $\ell' > 0$ will be a subsolution of this equation if it is such that (we impose $\ell' \leq 1$, which is no restriction, to simplify the writing of the second inequality):

$$0 < l' \leq Min\{1, \frac{\underset{V}{Inf}|r(\gamma')|}{\underset{V}{Sup}b}, \frac{\underset{M-V}{Inf}(a' + q_1')}{\underset{M-V}{Sup}(b + |r(\gamma'|}\}. \tag{4.33}$$

The function $\varphi_- \equiv \theta\ell' > 0$ is a subsolution of the original Lichnerowicz equation. It is always possible to choose the constant supersolution m large enough such that $\varphi_- \leq m$ on M.

2. If we know only that $\underset{U_1}{Inf} b > 0$, U_1 a subset of M, the problem is with the supersolutions. We try to conformally transform the metric γ to a metric with positive scalar curvature in V_1, with $M - U_1 \subset V_1$, by considering now the equation in V_1 and boundary condition, with k some positive number,

$$\Delta_\gamma \theta + k\theta = 0, \quad \theta \mid_{\partial V_1} = 1. \tag{4.34}$$

This equation implies that:

$$r(\gamma_1) \equiv \theta^{-\frac{n+2}{n-2}}(r(\gamma) + k), \quad \text{if} \quad \gamma_1 = \theta^{4/(n-2)}\gamma.$$

with $r(\gamma_1) > 0$ if

$$k > |r(\gamma)| \tag{4.35}$$

The problem 4.34 is of Fredholm type: it has a solution if the homogeneous problem, i.e. the equation 4.34 but with boundary condition $\theta_{\partial V_1} = 0$, has for unique solution 0, which is the case for $k > 0$, if k is smaller that the first eigenvalue. In this case the solution θ is positive. The conformally transformed equation (we extend θ to M, as above) has constant supersolutions, all numbers m_1 such that

$$m_1 \geq Max\{1, \frac{\underset{U_1}{Sup}(|r(\gamma_1)| + a_1 + q_1)}{\underset{U_1}{Inf} b}, \frac{\underset{V_1}{Sup}(a_1 + q_1)}{\underset{V_1}{Inf} \theta_1^{-(n+2)/(n-2)} r(\gamma_1)}\}, \tag{4.36}$$

A supersolution φ_+ of the original equation, arbitrarily large (but bounded), such that $\varphi_- \leq \varphi_+$, is deduced from the inverse conformal transformation.

a. If $r(\gamma) = 0$ the condition 4.35 is satisfied by any positive k.

b. If $r(\gamma) = -1$, the condition 4.35 is verified with a number k smaller than the first eigenvalue λ of the homogeneous problem . It holds that $\lambda = C_F^{-1}$ where C_F is the Friedrichs constant of the domain (V_1, γ), smallest number such that:

$$\int_{V_1} |u|^2 \mu_\gamma \leq C_F \int_{V_1} |Du|^2 \mu_\gamma, \quad \text{for all} \quad u \in \mathcal{D}(V_1). \tag{4.37}$$

■

Conclusion. Existence and non existence results cover all cases when b, that is τ, is a constant, since then either $\underset{M}{Inf} b > 0$ or $b \equiv 0$. A gap remains when τ is not constant and (M, γ) is in the negative Yamabe class.

Remark. The method can be applied in the presence of **unscaled sources**, with practically no change if $b - q_2 \geq 0$. If $b - q_2 < 0$ there can be solutions only if γ is in the positive Yamabe class (see a construction of sub and supersolutions in the case $n = 3$ in CB 1972). In the general case the discussion is more involved, but seems to offer no conceptual difficulty.

4.4 Scaled sources in L^p.

Though discontinuous energy sources are allowed by our hypothesis $q_1 \in L^\infty$, the following theorem, deduced from theorem 8.2, is useful in the presence of less regular sources and for the coupling with the momentum constraint with low regularity sources.

Theorem 4.12 *If (M, γ) is in the positive Yamabe class the theorem 4.10 and corollary 4.11 hold with hypothesis on a and q_1 weakened to $a, q_1 \in L^p$.*

The same result holds for the zero Yamabe class under the additional assumption $b \not\equiv 0$.

Proof. The problem in the case where $a + q_1$ is not bounded above is in the construction of a constant supersolution m. To construct a non constant supersolution, in the positive Yamabe case, we proceed as in Moncrief 1986 and CB-Moncrief 1994 (case $n = 2$). For simplicity we write up the physical case $n = 3$. We denote by $\varphi_0^4 \equiv y_0$ the positive number solution of the equation

$$\underline{b} y_0^3 + \underline{r} y_0^2 - \underline{q}_1 y_0 - \underline{a} = 0. \tag{4.38}$$

where $\underline{f}$ denotes the mean value of f on (M, γ) :

$$\underline{f} \equiv \frac{1}{Vol(M,\gamma)} \int_M f \mu_\gamma. \tag{4.39}$$

Such a number exists if $\underline{r} > 0$ or $\underline{r} = 0$ and $\underline{b} > 0$.

We define one function $v \in W_2^p$, with mean value zero on M, by solving the linear equation

$$\Delta_\gamma v = r\varphi_0 - a\varphi_0^{-7} - q_1\varphi_0^{-3} + b\varphi_0^5 \tag{4.40}$$

The function

$$\varphi_+ \equiv \varphi_0 + v - Inf v \geq \varphi_0, \quad \Delta_\gamma \varphi_+ \equiv \Delta_\gamma v, \tag{4.41}$$

is a supersolution because it holds that:

$$\Delta_\gamma \varphi_+ - f(.,\varphi_+) = r(\varphi_0 - \varphi_+) - a(\varphi_0^{-7} - \varphi_+^{-7}) - q_1(\varphi_0^{-3} - \varphi_+^{-3}) + b(\varphi_0^5 - \varphi_+^5) \tag{4.42}$$

hence if $r \geq 0$

$$\Delta_\gamma \varphi_+ - f(.,\varphi_+) \leq 0. \tag{4.43}$$

Since $\varphi_+ \geq \varphi_0 > 0$ we can choose a subsolution $\ell > 0$ such that $0 < \ell \leq \varphi_+$ and apply the theorem 8.2. ■

When $n = 3$ our result includes the case $p = 2$, i.e. γ, $\varphi \in H_2$.

5 Coupled system, τ =constant, scaled sources.

When τ is a constant the momentum constraint (scaled sources) does not depend on φ. We can therefore solve this linear system for X and insert afterwards the result in the semilinear equation for φ. To solve this Lichnerowicz equation we have supposed that $\gamma \in M_2^p$, $p > \frac{n}{2}$, and the coefficients $a, b, q_1 \in L^\infty$, except a and q_1 in the positive Yamabe case. While b and q_1 are given quantities, the function a depends on the given γ and B, and also on the unknown X, solution of the momentum constraint. We leave to the reader formulations of variants of the following existence theorem.

Theorem 5.1 *Let $\gamma \in M_2^p$. Let q_1 and J_1, L^2 orthogonal to the conformal Kiling fields of (M,γ), be given as well as $B \in W_1^p$, and τ =constant. Then the LCBY system with scaled sources admits a solution $X, \varphi \in W_2^p$, $\varphi > 0$, in the following cases*

1. (M,γ) is in the positive Yamabe class, $p > \frac{n}{2}$, $q_1, J_1 \in L^p$, $B \in W_1^p$ and $|J|_1 + q_1 \not\equiv 0$ on M.

2. (M,γ) is in the zero Yamabe class, $p > \frac{n}{2}$, q_1, $J_1 \in L^p$, $\tau \neq 0$ and $|J|_1 + q_1 \not\equiv 0$ on M.

3.(M,γ) is in the negative Yamabe class $p > n$, $q_1 \in L^\infty$, $J_1 \in L^p$, $B \in W_1^p$, $\tau \neq 0$.

The solution is unique, except if (M,γ) is in the zero Yamabe class and $|J|_1 + |B| + q_1 + \tau^2 \equiv 0$.

Proof. Under the hypothesis the momentum constraint 3.1 has a solution $X \in W_2^p$, since the vector F given by 3.2, here independent of φ and J_2, is in L^p and is L^2 orthogonal to the space of conformal Killing (CK) vector

fields of (M,γ). The tensor $A \equiv \mathcal{L}_{\gamma,conf}X + B$ is then in W_1^p. The function $a \equiv A.A$ is in L^p as soon as $p > \frac{n}{2}$.

1. and 2 The existence theorem 4.12, positive or zero Yamabe class, applies if $a + q_1 \not\equiv 0$. Since

$$D_i A^{ij} = J_1^j \tag{5.1}$$

we have $A \not\equiv 0$, hence $a \not\equiv 0$, if $J_1 \not\equiv 0$. Therefore $|J|_1 + q_1 \not\equiv 0$ implies $a + q_1 \not\equiv 0$.

3. If $p > n$ the tensor A is in C^0, hence the function a is also in $C^0 \subset L^\infty$. We apply the theorem 4.10 and its corollary 4.11. ■

6 Solutions with non constant TrK, or (and) unscaled sources.

We will show that there exists a whole neighbourhood of low regularity solutions with non constant[9] τ and (or) non zero unscaled sources near such a solution with constant τ and zero unscaled sources.

Lemma 6.1 *The mapping defined by:*

$$\Phi : (x,y) \mapsto \Phi(x,y) \equiv (\mathcal{H}(x,y), \mathcal{M}(x,y)), \tag{6.1}$$

$$x \equiv (\gamma, \tau, B, q_1, q_2, J_1, J_2), \quad y \equiv (\varphi, X) \tag{6.2}$$

$$\mathcal{H}(x,y) \equiv \Delta_\gamma \varphi - r(\gamma)\varphi + a(B,X)\varphi^{-\frac{3n-2}{n-2}} + q_1 \varphi^{-\frac{n}{n-2}} - (q_2 - \frac{n-2}{4n}\tau^2)\varphi^{\frac{n+2}{n-2}} \tag{6.3}$$

$$\mathcal{M}^i(x,y) \equiv (\Delta_{\gamma,conf}X)^i - \{\frac{n-1}{n}\varphi^{2n/(n-2)}\partial^i\tau + J_1^i + J_2^i \varphi^{2(n+2)/(n-2)} - D_j B^{ij}\} \tag{6.4}$$

is, if $p > \frac{n}{2}$, a differentiable map into the Banach space $\mathbf{B}_0 \equiv L^p \times {}^1\otimes L^p$ from the open set $\Omega \equiv \Omega_1 \times \Omega_2$ of the Banach space $\mathbf{B} \equiv \mathbf{B}_1 \times \mathbf{B}_2$, defined by[10]

$$\Omega_1 \equiv \mathbf{B}_1 \cap \{\gamma \in M_2^p\}, \quad \Omega_2 \equiv \mathbf{B}_2 \cap \{\varphi > 0\} \tag{6.5}$$

[9]There are cosmological spacetimes which do not admit surfaces of constant τ (Bartnik 1988)

[10]To help the reader follow our reasoning we denote by ${}^m \otimes W_s^p$ the space of W_s^p m-tensor fields.

$$\mathbf{B}_1 \equiv (W_2^p \times W_1^p \times (^2\otimes W_1^p) \times L^p \times L^p \times^1 \otimes L^p \times^1 \otimes L^p), \tag{6.6}$$

$$\mathbf{B}_2 \equiv W_2^p \times (^1\otimes W_2^p). \tag{6.7}$$

PROOF. The mapping Φ is continuous from Ω into $L^p \times^1 \otimes L^p$ because $\varphi \in W_2^p \subset C^0$, $\varphi > 0$, and $DB, \partial\tau, a, q_1, q_2, J_1, J_2 \in L^p$ as well as τ^2 and $r \equiv r(\gamma)$ when $p > \frac{n}{2}$, . It is differentiable on Ω because the linear mapping

$$\delta\Phi_{x,y} : (\delta x, \delta y) \mapsto \delta\Phi_{x,y}(\delta x, \delta y) \equiv (\delta\mathcal{H}_{x,y}(\delta x, \delta y), \delta\mathcal{M}_{x,y}(\delta x, \delta y)) \tag{6.8}$$

is a mapping from the Banach space $\mathbf{B}$ into the Banach space $L^p \times^1 \otimes L^p$ for each $(x, y) \in \Omega$. Indeed this linear mapping is given by:

$$\delta\mathcal{H}_{x,y}(\delta x, \delta y) \equiv \Delta_\gamma\delta\varphi - \{r + \frac{3n-2}{n-2}a\varphi^{-\frac{2n}{n-2}} + \frac{n}{n-2}q_1\varphi^{-2} + \frac{n+2}{n-2}(b-q_2)\varphi^{\frac{4}{n-2}}\}\delta\varphi$$

$$+\delta\gamma.D^2\varphi + \delta(\Gamma(\gamma)).D\varphi - \delta r\varphi + \delta a\varphi^{-\frac{3n-2}{n-2}} + \delta q_1\varphi^{-\frac{n}{n-2}} + (\delta q_2 - \delta b)\varphi^{\frac{n+2}{n-2}} \tag{6.9}$$

and

$$\delta\mathcal{M}_{x,y}(\delta x, \delta y) \equiv (\Delta_{\gamma,conf}\delta X)^i - (\frac{2(n-1)}{n-2}\varphi^{\frac{n+2}{n-2}}\partial^i\tau + \frac{2(n+2)}{n-2}\varphi^{\frac{n+6}{n-2}}J_2^i)\delta\varphi$$

$$+D_j\delta B^{ij} + (\delta(\Gamma(\gamma)).B)^i - \{\frac{n-1}{n}\varphi^{\frac{2n}{n-2}}\partial^i\delta\tau + \varphi^{\frac{2(n+2)}{n-2}}\delta J_2^i + \delta J_1^i\}. \tag{6.10}$$

Embeddings and multiplication properties give as in previous sections the announced result. ■

Theorem 6.2 *Let $(\bar{g}, \bar{K})$, $\bar{K} = \bar{B} + \mathcal{L}_{\bar{g},conf}\bar{X} + \frac{\bar{g}}{n}\bar{\tau}$ be a solution of the Einstein constraints on the compact n-manifold M, with Tr$K \equiv \bar{\tau}$ a constant, $\bar{g} \in M_2^p$, $\bar{X} \in W_2^p$, $\bar{B} \in W_1^p$, $p > \frac{n}{2}$ and scaled sources $\bar{q}_1$, $\bar{J}_1 \in L^p$ and no unscaled sources. Suppose $(M, \bar{g})$ admits no conformal Killing field. There is a neighbourhood of $\bar{\tau}, \bar{B}$, $\bar{q}_1$, $\bar{q}_2 = 0$, $\bar{J}_1$, $\bar{J}_2 = 0$ in $\mathbf{B}_1$ such that the Einstein constraints with non constant τ and (or) unscaled sources admit a solution (g, K) in a $W_2^p \times W_1^p$ neighbourhood of $(\bar{g}, \bar{K})$.*

PROOF. We have just proved that the mapping Φ defined by the left hand side of the LCBY equations is a differentiable map into a Banach space $\mathbf{B}_2$. Saying that the function $\bar{\varphi} = 1$ together with $\bar{X} \in W_2^p$ is a solution of the LCBY system with data $\bar{\gamma} \equiv \bar{g} \in M_2^p$, $\bar{\tau}$ a constant, $\bar{q}_1 \in L^p, , \bar{q}_2 = 0$, $\bar{B} \in W_1^p$, $\bar{J}_1 \in L^p$, $\bar{J}_2 = 0$ is to say that, with the notations of the previous lemma:

$$\Phi(\bar{x}, \bar{y}) = 0 \tag{6.11}$$

.By the implicit function theorem[11] there exists of a solution $y \equiv (\varphi, X) \in \Omega_2$, i.e. $\varphi \in W_2^p$, $\varphi > 0$, $X \in W_2^p$, when $x \equiv \{\gamma \in M_2^p$, $\tau \in W_1^p$, $B \in W_1^p$, q_1, q_2, J_1, $J_2 \in L^p\}$ is in a sufficiently small neighbourhood in the Banach space $\mathbf{B}$ of $\bar{x}$, under the condition that the partial derivative of Φ with respect to y at $(\bar{x}, \bar{y})$ is an isomorphism of Banach spaces, that is if the linear system

$$(\Delta_{\bar{g},conf} \delta X)^i = Y \tag{6.12}$$

and (recall that $\bar{\varphi} = 1$)

$$\Delta_{\bar{g}} \delta\varphi - \{\bar{r} + \frac{3n-2}{n-2}\bar{a} + \frac{n}{n-2}\bar{q}_1 + \frac{n+2}{n-2}\bar{b}\}\delta\varphi = \psi \tag{6.13}$$

has one and only one solution $\delta X \in^1 \otimes W_2^p$, $\delta\varphi \in W_2^p$ for each $Y \in^1 \otimes L^p$, $\psi \in L^p$. This result holds for 6.11 if $\bar{g}$ admits no conformal Killing field. It holds for 6.12 if the coefficient of $\delta\varphi$, which belongs to L^p, $p > \frac{n}{2}$ by hypothesis, is positive or zero, and not identically zero. It is obviously so when $\bar{r} > 0$. To treat general values of $\bar{r}$ we remark, as O'Murchada and York 1974 in their study of linearisation stability, that since $(\bar{g}, \bar{K})$ satisfies the hamiltonian constraint, hence the Lichnerowicz equation with $\bar{\varphi} = 1$, it holds that:

$$-\bar{r} + \bar{a} + \bar{q}_1 - \bar{b} = 0. \tag{6.14}$$

The linear equation 6.12 reads therefore

$$\Delta_{\bar{g}} \delta\varphi - \{2\bar{a} + \frac{2(n-1)}{n-2}\bar{q}_1 + \frac{4}{n-2}\bar{b}\}\delta\varphi = \psi \tag{6.15}$$

The coefficient of the lower order term $\delta\varphi$ is positive or zero, and not identically zero, if $\bar{a}$, $\bar{q}_1$, and $\bar{b}$ are not all identically zero. ■

The result does not extend in a straightforward way[12] to the case where $\bar{g}$ admits conformal Killing fields because $\mathcal{M}$ does not takes its values in the space of vectorfields orthogonal to such fields.

[11] See for instance CB and DeWitt I p.91.

[12] An analogous problem occurs in linearization stability (see Moncrief 1975)

7 Appendix A. Second order linear elliptic systems.

7.1 Linear systems

Let (M, e) be a smooth compact orientable riemannian manifold. Denote by ∂ the covariant derivative in the metric e. A second order linear differential operator from sections u of a tensor bundle E over (M, e) into sections of another such bundle F reads

$$Lu \equiv \sum_{k=0}^{2} a_k \partial^k u \tag{7.1}$$

with a_k a linear map from tensor fields to tensor fields given also by tensor fields over M. In local coordinates it reads:

$$(Lu)^A \equiv a_{2,B}^{A,ij} \partial_{ij}^2 u^B + a_{1,B}^{A,i} \partial_i u^B + a_{0,B}^{A} u^B \tag{7.2}$$

The principal symbol of the operator L at a point $x \in M$, for a covector ξ at x, is the linear map from E_x to F_x determined by the contraction of a_2 with $(\otimes\xi)^2$, represented in coordinates by the matrix

$$M_B^A(\xi) \equiv a_{2,B}^{A,ij} \xi_i \xi_j \tag{7.3}$$

The operator is said to be elliptic if for each $x \in M$ and $\xi \in T_x^* M$ its principal symbol is an isomorphism from E_x onto F_x for all $\xi \neq 0$, i.e. the determinant of the matrix M_B^A does not vanish. If there is a covering of M such that this determinant is uniformly bounded away from zero the system is said to be uniformly elliptic and the bound is called the ellipticity constant. The Laplace operator Δ_γ in a riemannian metric γ on M acting on scalar functions u has principal symbol $\gamma^{ij}\xi_i\xi_j$, it is elliptic.

The Sobolev spaces W_s^p on (M, e) are defined as closures of spaces of smooth tensor fields in the norm

$$||f||_{W_s^p} \equiv \{\int_M \sum_{0 \le k \le s} |\partial^k f|^p \mu_e\}^{\frac{1}{p}}, \tag{7.4}$$

where ∂, $|.|$ and μ_e denote the covariant derivative and the pointwise norm in the metric e.

We denote by C any positive number depending only on (M, e) : volume, Sobolev constant...

A metric γ is said to belong to M^p_σ if it is properly riemannian and $\gamma \in W^p_\sigma$. We will always suppose that $\sigma > \frac{n}{p}$, then $\gamma \in M^p_\sigma$ is continuous on M and M^p_σ is an open subset of W^p_σ. The contrevariant associate $\gamma^\#$ is also continuous and the volume elements μ_γ and μ_e are equivalent.

We recall the following theorem.

Theorem 7.1 *(Douglis and Nirenberg) Let Ω be a bounded open set of R^n, let $\tilde{L}$ be an homogeneous elliptic operator:*

$$\tilde{L}u \equiv a_m \tilde{\partial}^m u \tag{7.5}$$

where $\tilde{\partial}$ is the usual partial derivative and a_m is continuous and bounded on Ω, then the following estimate holds, for any $p > 1$:

$$\|u\|_{\tilde{W}^p_m} \leq C_{a_m}\{\|\tilde{L}u\|_{\tilde{L}^p} + ||u||_{\tilde{L}^p}\} \tag{7.6}$$

where $\tilde{W}^p_s$ denotes Sobolev spaces defined by the euclidean metric on Ω. The number C_{a_m} depends only on Ω the ellipticity constant, the C^0 norm of a_m and its modulus of continuity.

We will deduce from this theorem the following one[13].

Theorem 7.2 *Let (M, e) be a smooth compact orientable Riemannian manifold. Let*

$$Lu \equiv \sum_{k=0}^{2} a_k \partial^k u \tag{7.7}$$

be a second order elliptic operator on (M, e). Suppose the coefficients of L are such that:

$$a_2 \in W^p_2,\ a_1 \in W^p_1,\ a_0 \in L^p, \quad p > \frac{n}{2}.$$

Then

1. *The operator L is a continuous mapping $W^p_2 \to L^p$.*

[13]Lowering the regularity of the coefficients of the theorem proved in CB-Ch, it can be extended to higher order elliptic operators and possibly lower values of p for the solution, but we will need the restriction p>n/2 when working with the non linear Lichnerowicz equation and treat here only this case for simplicity.

2. The following estimate holds:

$$\|u\|_{W_2^p} \leq C_L\{\|Lu\|_{L^p} + ||u||_{L^1}\}. \tag{7.8}$$

The number C_L depends only on the norms of the $a_k's$ in their respective spaces and the ellipticity constant of a_2.

Corollary 7.3 *If morevover*

$$a_2 \in W_{2+s}^p,\ a_1 \in W_{1+s}^p,\ a_0 \in W_s^p,$$

then L is a continuous mapping $W_{s+2}^p \to W_s^p$ and for all $u \in W_{s+2}^p$ it holds that:

$$\|u\|_{W_{s+2}^p} \leq C_L\{\|Lu\|_{W_s^p} + ||u||_{L^1}\}. \tag{7.9}$$

with C_L a number depending only on the norms of the $a_k's$ and the ellipticity constant of a_2.

Proof. By the Sobolev embedding theorem we know that a_2 belongs to the Hölder space $C^{0,\alpha}$ since $W_2^p \subset C^{0,\alpha}$ if $p > \frac{n}{2}$. Therefore $a_2\partial^2 u \in L^p$ if $u \in W_2^p$.

If $a_1 \in W_1^p$ then $a_1\partial u \in L^p$ due to the continuous multiplication $W_1^p \times W_1^p \to L^p$ when $p > \frac{n}{2}$ and it holds that

$$||a_1\partial u||_{L^p} \leq C||a_1||_{W_1^p}||\partial u||_{W_1^p} \tag{7.10}$$

If $a_0 \in L^p$ then $a_0 u \in L^p$ whatever be u since $u \in C^{0,\alpha}$.

To prove the inequality 7.8 using the Douglis - Nirenberg theorem we proceed in two steps.

a. We treat the principal part by considering a covering of M by a finite number of charts with bounded domains Ω_i, and a partition of unity, functions $0 \leq \phi_i \leq 1$ with compact support in Ω_i such that $\sum_i \phi_i = 1$. We have $u = \sum u_i$, $u_i \equiv u\phi_i$. In any of the charts e is uniformly equivalent to the euclidean metric; we denote by $\tilde{\partial}$ the derivative in the euclidean metric (i.e. the usual partial derivative), by $\tilde{W}_s^p$ the associated Sobolev norm. For fields with support in a chart the norms W_s^p and $\tilde{W}_s^p$ are equivalent, they are uniformy equivalent for the various charts, since there is a finite number of them. The Douglis - Nirenberg theorem gives then the existence of a constant C_i, depending only on Ω_i and on a_2 through its $C^{0,\alpha}$ norm and the ellipticity constant of its representatives, such that

$$\|u_i\|_{W_2^p} \leq C_{i,a_2}\{\|\tilde{L}u_i\|_{L^p} + ||u||_{L^p}\}, \qquad \tilde{L}u_i \equiv a_2\tilde{\partial}^2 u_i. \tag{7.11}$$

The subadditivity of norms together with this inequality imply that, with C_{a_2} the maximum of the $C_i's$,

$$||u||_{W_2^p} \leq \sum_i ||u_i||_{W_2^p} \leq C_{a_2} \sum_i \{\|\tilde{L}u_i\|_{L^p} + ||u||_{L^p}\}. \tag{7.12}$$

On the other hand, we have an identity of the form, with $S(i)$ smooth coefficients linear in the connection of e :

$$\tilde{L}u_i \equiv a_2 \tilde{\partial}^2 u_i \equiv a_2\{\partial^2 u_i + S(i)\partial u_i\} \tag{7.13}$$

Using the Leibniz rule we obtain that

$$\partial u_i = u\partial\phi_i + \partial u\phi_i, \qquad \partial^2 u_i = \partial^2 u\phi_i + 2\partial u\partial\phi_i + u\partial^2\phi_i. \tag{7.14}$$

Since $0 \leq \phi_i \leq 1$ we deduce from 7.14 that

$$|a_2\partial^2 u_i| \leq |a_2\partial^2 u| + |a_2(2\partial u\partial\phi_i + u\partial^2\phi_i)|. \tag{7.15}$$

Combining previous inequalities we find that there exists a number C depending only on (M, e), such that

$$||\tilde{L}u_i||_{L^p} \leq C\{||a_2\partial^2 u||_{L^p} + ||a_2||_{C^0}||u||_{W_1^p}\}. \tag{7.16}$$

The inequality 7.12 implies then that:

$$||u||_{W_2^p} \leq C_{a_2}\{||a_2\partial^2 u||_{L^p} + ||u||_{L^p}\}. \tag{7.17}$$

b. We now consider the general linear operator L, with coefficients satisfying the given hypotheses:

$$Lu \equiv a_2\partial^2 u + a_1\partial u + a_0 u. \tag{7.18}$$

Since $a_2 \in W_2^p$, with $p > \frac{n}{2}$, it belongs also to a Hölder space $C^{0,\alpha}$, $0 < \alpha \leq 2 - \frac{n}{p}$, and it holds that, with C a Sobolev constant of (M, e),

$$||a_2||_{C^{0,\alpha}} \leq C||a_2||_{W_2^p}. \tag{7.19}$$

The inequality 7.17 implies that

$$||u||_{W_2^p} \leq C_{a_2}\{||Lu||_{L^p} + ||a_1\partial u + a_0 u||_{L^p} + ||u||_{W_1^p}\}. \tag{7.20}$$

To estimate $||a_1 \partial u||_{L^p}$ we use the Hölder inequality

$$||a_1 \partial u||_{L^p} \leq ||a_1||_{L^{p_1}} ||\partial u||_{L^q}, \quad \frac{1}{p_1} + \frac{1}{q} = \frac{1}{p}. \tag{7.21}$$

The Sobolev embedding theorem gives that

$$||a_1||_{L^{p_1}} \leq C ||a_1||_{W_1^p}, \quad p_1 = \frac{np}{n-p}. \tag{7.22}$$

Taking this value of p_1 gives that

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{p_1} = \frac{1}{n}. \tag{7.23}$$

and

$$||a_1 \partial u||_{L^p} \leq C ||a_1||_{W_1^p} ||\partial u||_{L^n}. \tag{7.24}$$

On the other hand:

$$||a_0 u||_{L^p} \leq ||a_0||_{L^p} ||u||_{L^\infty} \tag{7.25}$$

By the Gagliardo-Nirenberg interpolation inequality (see for instance Aubin p. 94, CB-DM II p.385) it holds that

$$||\partial^j u||_{L^q} \leq C \{ ||u||^{\lambda}_{W_2^p} ||u||^{1-\lambda}_{L^p} \tag{7.26}$$

with:

$$\lambda = \frac{n}{2} \{ \frac{j}{n} + \frac{1}{p} - \frac{1}{q} \} \tag{7.27}$$

if

$$0 < \lambda < 1. \tag{7.28}$$

Applying the equality 7.27 to the case $j = 1, q = n$ gives

$$0 < \lambda = \frac{n}{2p} < 1 \ \text{ since } \ p > \frac{n}{2} \tag{7.29}$$

Therefore, with this value of λ, it holds that:

$$||\partial u||_{L^n} \leq C ||\partial^2 u||^{\lambda}_{L^p} ||u||^{1-\lambda}_{L^p}. \tag{7.30}$$

The same procedure applied to the case $j = 1, p = q, \lambda = \frac{1}{2}$ gives that

$$||\partial u||_{L^p} \leq C ||\partial^2 u||^{\frac{1}{2}}_{L^p} ||u||^{\frac{1}{2}}_{L^p}. \tag{7.31}$$

The interpolation inequality 7.26 with $q = \infty$ and $j = 0$ gives, with again $\lambda = \frac{n}{2p}$:

$$||u||_{L^\infty} \leq C||\partial^2 u||_{L^p}^{\lambda}||u||_{L^p}^{1-\lambda}. \tag{7.32}$$

For any $\varepsilon > 0$, by elementary calculus, it holds that

$$||\partial^2 u||_{L^p}^{\lambda}||u||_{L^p}^{1-\lambda} \leq \lambda\varepsilon||\partial^2 u||_{L^p} + \frac{1}{(1-\lambda)\varepsilon^{\lambda/1-\lambda}}||u||_{L^p} \tag{7.33}$$

hence there exists a number C such that

$$||a_1\partial u + a_0 u||_{L^p} \leq C\{||a_1||_{W_1^p} + ||a_0||_{L^p}\}\{\varepsilon||u||_{W_2^p} + C_\varepsilon||u||_{L^p}\} \tag{7.34}$$

Using this inequality together with 7.20 we see that we can choose $\varepsilon > 0$ small enough, depending on the bound of $||a_1||_{W_1^p} + ||a_0||_{L^p}$ and C_{a_2} such that the inequality

$$\|u\|_{W_2^p} \leq C_L\{\|Lu\|_{L^p} + ||u||_{L^p}\} \tag{7.35}$$

is satisfied. But by the definition of integrals and norms it holds that

$$||u||_{L^p} \leq ||u||_{L^\infty}^{(p-1)/p}||u||_{L^1}^{1/p} \leq C||u||_{W_2^p}^{(p-1)/p}||u||_{L^1}^{1/p} \tag{7.36}$$

Therefore if $p > \frac{n}{2}$, for any $\varepsilon > 0$ there exists ε and C_ε such that:

$$||u||_{L^p} \leq ||u||_{W_2^p}^{(p-1)/p}||u||_{L^1}^{1/p} \leq \varepsilon||u||_{W_2^p} + C_\varepsilon||u||_{L^1}, \tag{7.37}$$

we can again choose ε so that the announced inequality holds. ■

PROOF. of corollary. This regularity statement can be proved by standard recursive arguments. ■

Theorem 7.4 *Under the hypothesis of the theorem 7.2 for $s = 0$, and its corollary 7.3 in the case $s > 0$, it holds that:*

1. The operator L maps W_{2+s}^p into W_s^p with finite dimensional kernel and closed range.

2. If L is injective on W_{2+s}^p, then there is a number C_L such that for each u in W_{s+2}^p the following inequality holds:

$$\|u\|_{W_{2+s}^p} \leq C_L\|Lu\|_{W_s^p}\ . \tag{7.38}$$

3. If the formal adjoint $^*L$ of L satisfies the same hypothesis as L and is injective, then L is surjective from W_{2+s}^p onto W_s^p, hence an isomorphism if also injective.

PROOF. It follows usual lines. ■

7.2 Isomorphism theorem for the Poisson operator.

The Poisson operator in a riemannian metric γ on (M,e) acts on scalar functions u and reads, with S^h_{ij} the difference of the connections of e and γ :

$$\triangle_\gamma u - au \equiv \gamma^{ij}(\partial^2_{ij}u + S^h_{ij}\partial_h u) - au. \tag{7.39}$$

We suppose that $\gamma \in M^p_2$, $p > \frac{n}{2}$, then $\gamma \in C^{0,\alpha}$. We have denoted by M^p_2 the open subset of symmetric 2-tensors in W^p_2 which are properly riemannian metrics, namely such that they admit a positive ellipticity constant, $\underset{\cup M_I}{Inf}\det(\gamma_{ij}) > 0$, where γ_{ij} are the components of γ in a finite number of charts M_I covering M. The coefficient $a_2 \equiv \gamma^\#$, the contrevariant tensor associated to γ, is then in $C^{0,\alpha}$ with a positive ellipticity constant. The coefficient $a_1 \equiv \gamma^{ij}S_{ij}$ is in W^p_1 because $\partial\gamma\partial\gamma \in L^p$ if $\gamma \in W^p_2$, $p > \frac{n}{2}$. The hypothesis of the theorem 7.2 are satisfied if $a \in L^p$. Hence:

Lemma 7.5 *If $\gamma \in M^p_2$, $p > \frac{n}{2}$, the Poisson operator is a continuous mapping $W^p_2 \to L^p$ if $a \in L^p$.*

We now prove a uniqueness lemma[14].

Lemma 7.6 *Let $\gamma \in M^p_2$, $p > \frac{n}{2}$, $a \in L^p$, the Poisson operator $\triangle_\gamma - a$ is injective on W^p_2 if $a \geq 0$, $a \not\equiv 0$ (i.e. $a > 0$ on a subset of M of positive measure).*

Proof. The proof for C^2 functions results from the maximum principle if the coefficients are bounded. In the more general case one proceeds as follows: on a compact manifold, if u_n is a C^2 function and $\gamma_n \in C^1$, the following identity is obtained by a straightforward integration by parts

$$\int_M u_n(\Delta_{\gamma_n}u_n)\mu_{\gamma_n} \equiv -\int_M (\gamma^{ij}_n\partial_i u_n\partial_j u_n)\mu_{\gamma_n} \tag{7.40}$$

This identity applied to a C^2 solution of $\Delta_{\gamma_n}u_n - au_n = 0$, implies that $\partial_i u_n \equiv 0$ on M, and $u_n = 0$ on the subset $a > 0$. Hence u_n =constant, and $u_n \equiv 0$.

Suppose now that $\gamma \in M^p_2$, $p > \frac{n}{2}$, and $u \in W^p_2$. Since C^2 is dense in W^p_2 we can approximate γ and u, and a by smooth sequences γ_n, u_n converging

[14]The conditions p>n/2 for γ and for a are not necessary, but they are sufficient.

to γ and u, respectively in M_2^p, W_2^p. We already know by the lemma 7.5 that $\Delta_{\gamma_n} u_n$ converges in L^p to $\Delta_\gamma u$. Since the measures μ_{γ_n} and μ_γ are equivalent to μ_e and the continuous embedding $W_2^p \times L^p \to L^1$ always holds as well as $W_1^p \times W_1^p \subset L^1$ if $p > \frac{n}{2}$, the integrals on both sides of 7.30 converge to limits, and it holds that:

$$\int_M u(\Delta_\gamma u - au)\mu_\gamma \equiv -\int_M (\gamma^{ij}\partial_i u \partial_j u + au^2)\mu_\gamma \tag{7.41}$$

from which follows $u \equiv 0$ if $\Delta_\gamma u - au = 0$ and $a \geq 0, a \not\equiv 0$. ■

Theorem 7.7 *The Poisson operator $\triangle_\gamma - a$ on scalar functions in a metric γ on a smooth compact riemannian manifold (M, e), with $\gamma \in M_2^p$, $p > \frac{n}{2}$, $a \in L^p$, is an isomorphism from W_2^p onto L^p if $a \geq 0$, $a \not\equiv 0$.*

It is an isomorphism $W_{s+2}^p \to W_s^p$, if in addition $a \in W_s^p$.

Proof. Under the given hypothesis the Poisson operator is selfadjoint relatively to the metric γ : two integrations by part for smooth enough γ, u and v, and taking limits, show that under the given hypothesis, as in the proof of the lemma 7.6, the following identity holds for $u, v \in W_2^p$:

$$\int_M v(\Delta_\gamma u - au)\mu_\gamma \equiv \int_M u(\Delta_\gamma v - av)\mu_\gamma. \tag{7.42}$$

The isomorphism theorem 7.4 applies therefore to our Poisson operator. ■

7.3 Generalized maximum principle.

One knows by the classical maximum principle[15] that if a solution $u \in C^2$ of the inequality with bounded coefficients and $a \geq 0$

$$\triangle_\gamma u - au \leq 0, \quad [\text{respectively} \quad \triangle_\gamma u - au \geq 0]$$

attains a minimum $\lambda \leq 0$ [respectively a maximum $\lambda \geq 0$] at a point of M then $u \equiv \lambda$ on[16] M. Therefore $\triangle_\gamma u - au \leq 0$ on M implies $u \geq 0$ on M if $a \geq 0$ on M. We will need to obtain low regularity solutions the following generalization of the maximum principle.

[15] Protter and Weinberger 1967, p.64.

[16] Also if S is a bounded domain of M with smooth boundary a minimum ≤ 0 [respectively a maximum ≥ 0]must be attained on the boundary.

Lemma 7.8 *If $u \in W_2^p$, $p > \frac{n}{2}$ satisfies the equation*

$$\Delta_\gamma u - au = -f \tag{7.43}$$

with $\gamma \in M_2^p$, $a \in L^p$, $a \geq k$, with $k > 0$ a number , and $f \geq 0$, then $u \geq 0$ on M.

Proof. The lemma holds by the classical maximum principle if $u \in C^2$, $\gamma \in M_3^p$ $a \in C^0$. We approximate $\gamma \in M_2^p$ by $\gamma_n \in M_3^p$, a by $a_n \in C^0$, $a_n \geq k$ and $f \in L^p$ by $f_n \in C^0$, $f_n \geq 0$. We know by the isomorphism theorem that each equation

$$L_n \equiv \Delta_{\gamma_n} u_n - a_n u_n = -f_n$$

has one solution $u_n \in W_4^p \subset C^2$, and $u_n \geq 0$. We know that there exists a number C_L depending only on the W_2^p norm of γ_n, hence of γ, and the L^p norm of a_n, hence of a, such that:

$$||u_n||_{W_2^P} \leq C_L\{||f_n||_{L^p} + ||u_n||_{L^1}\}. \tag{7.44}$$

We also know that, since $L_n \equiv \Delta_{\gamma_n} - a_n$ is injective on W_2^p, there exists a number C_{L_n} such that:

$$||u_n||_{W_2^P} \leq C_{L_n}||f_n||_{L^p}, \tag{7.45}$$

however the number C_{L_n} does not depend only on the W_2^p and L^p norms of γ_n and a_n, hence is not a priori independent of n. In order to obtain our positivity result by a continuity argument we must estimate C_{L_n} independently of n. We proceed as follows.

The equality 7.41 shows that a solution $u \in W_2^p$, $p > \frac{n}{2}$, of 7.43 satisfies the integral equality

$$-\int u(\Delta_\gamma u - au)\mu_\gamma = \int_M \gamma^{ij}\partial_i u \partial_j u + au^2)\mu_\gamma = \int_M uf\mu_\gamma.$$

which implies if $a \geq k > 0$

$$\int_M u^2 \mu_\gamma \leq k^{-1} \int_M uf\mu_\gamma \tag{7.46}$$

This inequality and the uniform equivalence through a constant C_γ (depending only on the W_2^p norm of γ and its ellipticity constant) of the quadratic forms γ and e, together with the Hölder inequality give:

$$||u||^2_{L^2} \leq C_{\gamma,k}||u||_{L^q}||f||_{L^{q'}}, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad C_{\gamma,k} = C_\gamma k^{-1} \tag{7.47}$$

We have supposed $f \in L^p$.

Suppose first that $p \geq 2$. We have then, since M is compact, $f \in L^2$ with

$$||f||_{L^2} \leq C||f||_{L^p} \tag{7.48}$$

We choose then $q' = 2$, $q = 2$ and we deduce from 7.46

$$||u||_{L^1} \leq C||u||_{L^2} \leq C_{\gamma,k}||f||_{L^p}. \tag{7.49}$$

Hence for a solution u of a Poisson equation with coefficients $\gamma \in M_2^p$, $a \in L^p$, $p > \frac{n}{2}$, $a \geq k > 0$ an inequality

$$||u||_{W_2^P} \leq C_{L,k}||f||_{L^p}, \tag{7.50}$$

where $C_{L,k}$ depends only on the W_2^p and L^p norms of γ and a, and the value of k.

Suppose that $p < 2$ (inequality compatible with $p > \frac{n}{2}$ only if $n < 4$). We then take $q' = p$, but we have $q > 2$, since $q^{-1} = 1 - p^{-1}$. We use the equality

$$||u||_{L^q} \leq C||u||_{W_2^p}^{(q-2)/q}||u||_{L^2}^{2/q} \tag{7.51}$$

to obtain that

$$||u||_{L^2}^{2-\frac{2}{q}} \leq C||u||_{W_2^p}^{(q-2)/q}||f||_{L^p} \tag{7.52}$$

hence

$$||u||_{L^2} \leq C||u||_{W_2^p}^{(q-2)/2(q-1)}||f||_{L^p}^{q/2(q-1)} \leq C\varepsilon||u||_{W_2^p} + C_\varepsilon||f||_{L^p} \tag{7.53}$$

from which follows again an inequality of the type 7.50.

We apply a similar inequality to the difference $u_n - u_{n-1}$ which satisfies the equation:

$$\Delta_{\gamma_n}(u_n - u_{n-1}) - a_n(u_n - u_{n-1}) = F_n \tag{7.54}$$

with

$$F_n \equiv (\Delta_{\gamma_n} - \Delta_{\gamma_{n-1}})u_{n-1} - (f_n - f_{n-1}) + (a_n - a_{n-1})u_{n-1}.$$

The expression of Δ_γ as linear in the second and first derivatives of u, the usual embedding and multiplication properties, and an inequality of the type 7.50 satisfied by u_{n-1}, show that $F_n \in L^p$ with an L^p bound of the form

$$||F_n||_{L^P} \leq C_{\gamma,f,k}\{||\gamma_n - \gamma_{n-1}||_{W_\sigma^p} + ||f_n - f_{n-1}||_{L^p} + ||a_n - a_{n-1}||_{L^p}\}. \tag{7.55}$$

where $C_{\gamma,f,k}$ denotes any positive number depending only on k, the W_2^p norm of γ, and the L^p norms of a and f. The inequality 7.50 applied to 7.54 gives therefore an inequality of the form

$$||u_n - u_{n-1}||_{W_2^p} \leq C_{\gamma,f,k}\{||\gamma_n - \gamma_{n-1}||_{W_2^p} + ||f_n - f_{n-1}||_{L^p} + ||a_n - a_{n-1}||_{L^p}\}.$$

We deduce from this inequality that the sequence u_n converges in W_2^p, hence in C^0, to a function $u \in W_2^p$ when f_n converges to f and γ_n to γ. This function u is the unique solution of the Poisson equation, and since $u_n \geq 0$, the same is true of u[17]. ■

8 Appendix B. Equation $\Delta_\gamma \varphi = f(x, \varphi)$.

We consider equations of the form, with φ a scalar function

$$\Delta_\gamma \varphi = f(., \varphi). \tag{8.1}$$

where $f : (x, y) \mapsto f(x, y)$ is a function on $M \times I$, with I an interval of R, smooth in y. To be specific, and in view of application to the Lichnerowicz equation, we suppose that f is a finite sum:

$$f(x, y) \equiv \sum_{I=1,\ldots N} a_I(x) y^{P_I}, \tag{8.2}$$

where the exponents P_I are given real numbers. Hence f is a smooth function of y if $y \geq \ell > 0$.

It is to solve this semi-linear equation that we have supposed $p > \frac{n}{2}$. The space W_2^p is then an algebra, φ^P is in $W_2^p \subset C^0$ for any P if it is so of φ and if φ is positive.

To solve 8.1 we apply a method[18] of successive iterations obtained by resolution of linear problems, whose bounds and convergence are obtained through the use of sub and super solutions.

[17] Recall that it is to obtain a family of smooth functions u$_n$ approaching u that we had to introduce the positive k. Generalized maximum principles can probably be obtained by other methods.

[18] The solution of equations of the indicated type on compact manifolds was first obtained, in Holder spaces, by using the Leray-Shauder degree theory, together with bounds by sub and super solutions in CB and Leray 1972. It was applied to the constraints by CB 1972 (unscaled sources), O'Murchada and York 1974 (scaled sources), extended to H$_s$ spaces in CB 1975. The iteration method (Kazdan and Warner 1985) was applied to the constraints by Isenberg 1987. In the following we weaken the regularity hypothesis previously made.

definition 8.1 *A W_2^p function φ_- is called a* ***subsolution*** *of $\Delta_\gamma \varphi = f(.,\varphi)$ if it is such that on*[19] *M*

$$\Delta_\gamma \varphi_- \geq f(.,\varphi_-). \tag{8.3}$$

An W_2^p function φ_+ is called a supersolution if

$$\Delta_\gamma \varphi_+ \leq f(x,\varphi_+). \tag{8.4}$$

Theorem 8.2 *(existence) The equation $\Delta_\gamma \varphi = f(.,\varphi)$ on the compact manifold (M,γ) admits a solution $\varphi \in W_2^p$, $p > \frac{n}{2}$, if:*

a. $\gamma \in M_2^p$, $p > \frac{n}{2}$, $a_I \in L^p$, I=1,...,N.

b. The equation admits a subsolution φ_- and a supersolution φ_+, both in W_2^p, $0 < \ell \leq \varphi_- \leq \varphi_+ \leq m$. The solution is such that $\varphi_- \leq \varphi \leq \varphi_+$.

Proof. The proof follows the same lines as the standard proof, but uses the generalized maximum principle (lemma 7.8). We define successive iterates by solution of the elliptic equation, linear in φ_n when φ_{n-1} is known,

$$\Delta_\gamma \varphi_n - a\varphi_n = f(.,\varphi_{n-1}) - a\varphi_{n-1}, \tag{8.5}$$

with $a \in L^p$, $a \geq k$, k a strictly positive number to be determined later, introduced now to allow the application of the theorem 7.7.

We take $\varphi_0 = \varphi_-$ to start the iteration, i.e. we define φ_1. by solving the equation

$$\Delta_\gamma \varphi_1 - a\varphi_1 = f(.,\varphi_-) - a\varphi_-, \tag{8.6}$$

due to the hypothesis made on γ, a, f and φ_-, this solution φ_1 is in W_2^p. Moreover we have by the definitions of φ_1 and φ_-

$$\Delta_\gamma(\varphi_1 - \varphi_-) - a(\varphi_1 - \varphi_-) \leq 0 \tag{8.7}$$

The conclusion $\varphi_1 - \varphi_- \geq 0$ on M follows from the generalized maximum principle.

We now use the definition of φ_+ to obtain:

$$\Delta_\gamma(\varphi_+ - \varphi_1) - a(\varphi_+ - \varphi_1) \leq f(x,\varphi_+) - f(x,\varphi_-) - a(\varphi_+ - \varphi_-) \tag{8.8}$$

which gives using the mean value theorem (CB-DM I p.78)

[19] Inequality satisfied for functions in L^p, i.e. almost everywhere

$$\Delta_\gamma(\varphi_+-\varphi_1)-a(\varphi_+-\varphi_1) \leq (\varphi_+-\varphi_-)\int_0^1 \{f'_\varphi(x,\varphi_-+t(\varphi_+-\varphi_-))-a\}dt \quad (8.9)$$

It is sufficient to choose the function a such that

$$a = Sup_{l\leq y\leq m}|f'_y(.,y)| + k, \quad k > 0 \tag{8.10}$$

to have on M that $a \geq k > 0$ and

$$\Delta_\gamma(\varphi_+ - \varphi_1) - a(\varphi_+ - \varphi_1) = F \leq 0, \tag{8.11}$$

since on M it holds that $0 < \ell \leq \varphi_- + t(\varphi_+ - \varphi_-) \leq m$. For example if the function f is given by 8.2, then:

$$|f'_y(.,y)| \leq \sum_{I=1,...N} |a_I P_I| y^{P_I - 1} \leq \sum_{I_+} |a_{I_+} P_{I_+}| m^{P_{I_+}-1} + \sum_{I_-} |a_{I_-} P_{I_-}| \ell^{P_{I_-}-1} + |a_{I_0}| \tag{8.12}$$

with $P_{I_+} > 1$, $P_{I_-} < 1$, $P_{I_0} = 1$. Under the hypothesis made on a_I the maximum principle holds, hence $\varphi_1 \leq \varphi_+$ on M.

Suppose that there exists $\varphi_m \in W^p_2$, $m = 1, ..., n-1$, such that on M

$$\varphi_- \leq \varphi_m \leq \varphi_+ \tag{8.13}$$

then φ_n exists on M, solution of 8.5, also in W^p_2. Suppose moreover that

$$\varphi_{n-1} - \varphi_{n-2} \geq 0 \tag{8.14}$$

Then the equations satisfied by φ_n and φ_{n-1} imply

$$\Delta_\gamma(\varphi_n - \varphi_{n-1}) - a(\varphi_n - \varphi_{n-1}) = (b_n - a)(\varphi_{n-1} - \varphi_{n-2}) \tag{8.15}$$

where b_n is the function in L^p

$$b_n \equiv \int_0^1 f'_\varphi(., \varphi_{n-2} + t(\varphi_{n-1} - \varphi_{n-2})dt. \tag{8.16}$$

It results from the definition of a that:

$$b_n - a \leq -k < 0 \tag{8.17}$$

this inequality together with $\varphi_{n-1} \geq \varphi_{n-2}$ insures that the right hand side of (8.15) is negative, hence $\varphi_- \leq \varphi_{n-1} \leq \varphi_n$. To show that the sequence is bounded above by φ_+ we write again an inequality of the form

$$\Delta_\gamma(\varphi_+ - \varphi_n) - a(\varphi_+ - \varphi_n) \leq f(\varphi_+) - f(\varphi_{n-1}) - a(\varphi_+ - \varphi_{n-1}) \tag{8.18}$$

$$\leq (b_n - a)(\varphi_+ - \varphi_{n-1}) \leq 0,$$

from which the conclusion follows. We have proved the existence of the sequence $\varphi_n \in W_2^p$, with $0 \leq \varphi_- \leq \varphi_n \leq \varphi_+$ hence $||\varphi_n||_{L^p}$ uniformly bounded by $||\varphi_+||_{L^p}$. Due to the defining equation 8.5 $||\varphi_n||_{W_2^p}$ is also uniformly bounded. Since functions in W_2^p are equicontinuous we can extract (Ascoli - Arzela theorem) from the sequence φ_n a subsequence, $\tilde{\varphi}_n$, which converges in C^0 norm, to a function $\varphi \in C^0$. On the other hand the sequence φ_n of continuous and positive functions is increasing and bounded above by φ_+, hence it is pointwise convergent to a funtion φ, which coincides with the previously obtained φ. The whole sequence (not only a subsequence) is therefore convergent.

We show that the limit is in fact in W_2^p by using the elliptic estimate applied to the equation 8.15

$$\| \varphi_n - \varphi_{n-1} \|_{W_2^p} \leq C_{\gamma,a} ||(b_n - a)(\varphi_{n-1} - \varphi_{n-2}) \|_{L^p} \tag{8.19}$$

which implies:

$$\| \varphi_n - \varphi_{n-1} \|_{W_2^p} \leq C_{\gamma,a} ||(b_n - a)||_{L^p} ||\varphi_{n-1} - \varphi_{n-2}) \|_{C^0}, \tag{8.20}$$

The sequence φ_n converges in W_2^p (to φ by uniqueness of limits) since it converges in C^0, and the $b_n - a$ are uniformly bounded in L^p norm. The limit satisfies the equation in L^p sense.

The solution we have constructed depends on the choice of the initial φ_0. We have taken φ_- as an initial φ_0. We could have taken φ_+ and construct a decreasing and bounded below sequence which converges to a limit Ψ. We cannot in the procedure used start from an arbitrary φ_0, with $\varphi_- \leq \varphi_0 \leq \varphi_+$, because we will not know if φ_1 satisfies the same inequalities. A uniqueness theorem is easy to prove in the case where $y \mapsto f(., y)$ is monotonically increasing. A more general uniqueness theorem (4.4) holds for the Lichnerowicz equation. ∎

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